REGULARIZATION OF THE SCHEME FOR SOLVING REVERSE HEAT-CONDUCTION PROBLEMS

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A procedure is developed for determining the thermal fluxes from the stable solution to a reverse heat-conduction problem, and the computational efficiency of this scheme is evaluated.

In [1] the author has constructed algorithms for a stable determination of transient thermal fluxes from temperature measurements at one point of a semiinfinite body or a flat plate with a constant thermal diffusivity. Let us evaluate the efficiency of the proposed regularization in solving a reverse heat-conduction problem, using as an example a semiinfinite body with a stationary boundary and a zero initial temperature field. For this case, according to [2], we write the recurrence relation for the thermal flux:

$$\bar{q}_n = \frac{1}{\varphi_n^n}\left[\Theta_n(x_1) - \sum_{i=1}^{n-1} \bar{q}_i \varphi_i^n\right], \quad n = 1, 2, \ldots, m, \tag{1}$$

where $\Theta_n(x_1) = (1/\lambda_0)\int_0^{T_n(x_1)} \lambda(T)dT$ represents the model temperature at the n-th instant of time at point $x = x_i$,

$$\varphi_i^n = 2\sqrt{\Delta\text{Fo}}\left\{\sqrt{n-i+1}\, i\Phi^*\left[\frac{1}{2\sqrt{\Delta\text{Fo}\,(n-i+1)}}\right] - \sqrt{n-i}\, i\Phi^*\left[\frac{1}{2\sqrt{\Delta\text{Fo}\,(n-i)}}\right]\right\};$$

$$i\Phi^*[u] = \frac{1}{\sqrt{\pi}}\exp[-u^2] - u\left(1 - \frac{2}{\sqrt{\pi}}\int_0^u \exp[-\eta^2]\,d\eta\right);$$

$$\bar{q}_i = \frac{x_1}{\lambda_0}\left(\frac{q_i + q_{i-1}}{2}\right); \quad \Delta\text{Fo} = \frac{a\Delta\tau}{x_1^2}; \quad \Delta\tau = \frac{\tau_m}{m}.$$

We stipulate the following norms:

$$\|\Theta\| = \left[\sum_n \Theta_n^2\right]^{1/2}, \quad \|\bar{q}'\| = \left[\sum_i \frac{(\bar{q}_{i+1} - \bar{q}_i)^2}{\Delta\tau}\right]^{1/2},$$

$$\|A_{\Delta\tau}\| = \left[\sum_{n,i} (\varphi_i^n)^2\right]^{1/2}$$

and consider the A. N. Tikhonov regularizing functional for (1)

$$\Phi_{\Delta\tau}^{\alpha}[\bar{q}, \Theta] = \|A_{\Delta\tau}\bar{q} - \Theta\|^2 + \alpha\|\bar{q}'\|^2, \quad \alpha > 0. \tag{2}$$

Minimizing (2) with respect to $\bar{q}$, with the initial and the boundary condition

$$\bar{q}'(0) \sim \frac{\bar{q}_1 - \bar{q}_0}{\Delta\text{Fo}} = C_1, \quad \bar{q}'(\tau_m) \sim \frac{\bar{q}_{m+1} - \bar{q}_m}{\Delta\text{Fo}} = C_2,$$

we obtain a system of linear algebraic equations with a symmetrical positive-definite matrix

$$\sum_{l=1}^{m} a_{lk}\bar{q}_l = f_k, \quad k = 1, 2, \ldots, m,$$

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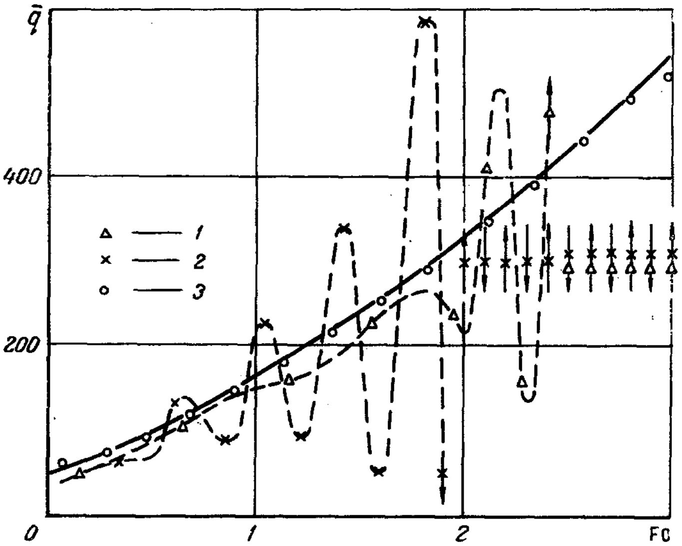

Fig. 1. Determination of thermal fluxes $\bar{q}$ (°C) at x_1 = 0.6 mm with ΔFo = 0.06: by the direct method with $\delta_0 \sim 0.001\Theta_{max}$ (3), exact solution (solid line).

where

$$a_{lk} = \Delta\text{Fo}^2 \sum_{n=l}^{m} \varphi_k^n \varphi_l^n, \; l \geqslant k+2;$$

$$a_{lk} = \Delta\text{Fo}^2 \sum_{n=l}^{m} \varphi_k^n \varphi_l^n - \alpha, \; l = k+1;$$

$$a_{lk} = \Delta\text{Fo}^2 \sum_{n=l}^{m} (\varphi_l^n)^2 + 2\alpha, \; l \neq 1, m;$$

$$a_{ll} = \Delta\text{Fo}^2 \sum_{n=l}^{m} (\varphi_l^n)^2 + \alpha, \; l = 1, \; m;$$

$$f_1 = \sum_{n=l}^{m} b_{1n}\Theta_n - \alpha C_1 \Delta\text{Fo};$$

$$f_k = \sum_{n=k}^{m} b_{kn}\Theta_n, \;\; k \neq 1, \; m; \;\; b_{kn} = \Delta\text{Fo}^2 \varphi_k^n; \; f_m = b_{mm}\Theta_m + \alpha C_2 \Delta\text{Fo}.$$

System (3) with a given value of parameter α is best solved by the square-root method. The choice of the closest approximation to q (the choice of α) will be made by the Tikhonov–Glasko method of the quasi-optimum parameter [3] ($\alpha_{j+1} = \varkappa\alpha_j, \; \varkappa > 0$)

$$\min_{\alpha} \{ \Delta_1 = \max_{\tau} | \bar{q}_{\alpha_{j+1}} - \bar{q}_{\alpha_j} | \}, \; \alpha = \alpha_{qo}; \tag{4}$$

the remainder principle

$$\min_{\alpha} \left\{ \Delta_2 = \left| \left\{ \sum_{n=1}^{m} \left[\sum_{i=1}^{n} \varphi_i^n \bar{q}_{i\alpha} - \Theta_{\delta n} \right]^2 \Delta\text{Fo} \right\}^{1/2} - \delta_{L_2} \right| \right\}, \; \alpha = \alpha_r, \tag{5}$$

and the value of the regularizing functional in the regularized solution

$$\min_{\alpha} \{ \Phi_{\Delta\tau}^{\alpha} [\bar{q}_\alpha] - \delta_{L_2} \}, \; \alpha = \alpha_{\text{reg}}. \tag{6}$$

The last two methods were proposed and explained by V. A. Morozov [4, 5].

In accordance with the developed algorithms, a program has been set up in the ALGOL-60 language and various methodical examples were calculated on an M-220 computer. Some results are shown here.

This already known formula

$$\bar{q}(\text{Fo}) = 48 + 88\text{Fo} + 27\text{Fo}^2$$

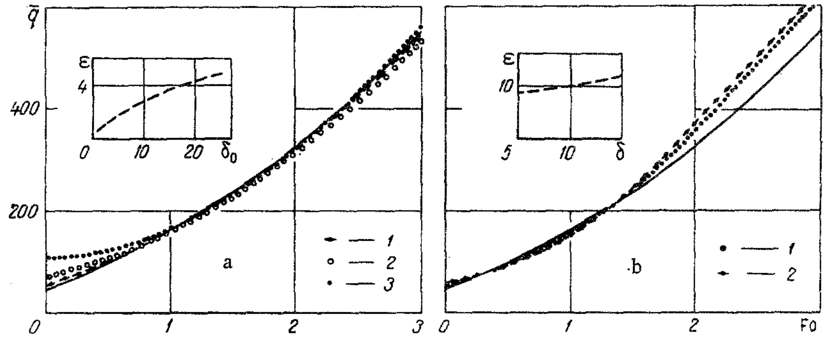

Fig. 2. Thermal fluxes $\bar{q}$ (°C): at (a) $x_1 = 0.6$ mm with ΔFo = 0.06 and $\delta_0 = 0.05\Theta_{max}(18°C)$ (1), $0.25\Theta_{max}(88°C)$ (2), $0.25\Theta_{max}(88°C)$ and $C_1 = 0$ (3); (b) with a normal distribution of errors in the original temperature readings, at $x_1 = 2.4 \cdot 10^{-3}$ m with ΔFo = 0.004 and $\delta_0 = 0.05\Theta_{max}$ (1), $0.15\Theta_{max}$ (2). Solid lines represent the exact solution, dashed lines represent the accuracy of restoring the thermal flux curve at known values of C_1 and C_2. Parameters ε and δ both in %.

was being restored on the basis of temperature values at a given point inside a body with a sufficiently low conductivity ($\lambda = 1.3 \cdot 10^{-4}$ kW/m·°C) and thermal diffusivity ($a = 1.2 \cdot 10^{-7}$ m^2/sec). The input function $\Theta(x_1)$ was determined from the solution to the forward heat-conduction problem, on the basis of a given thermal flux. The values of $\Theta_n(x_1)$ which had been computed with an error smaller than 10^{-2}–10^{-3}°K were tentatively accepted as exact data for solving the reverse heat-conduction problem.

In Fig. 1 is shown an example of a $\bar{q}$(Fo) curve computed with exact data for $\alpha = 0$, $x_1 = 0.6 \cdot 10^{-3}$ m, and ΔFo = 0.06. Beginning at Fo ~ 2, evidently, the values for $\bar{q}$ differ appreciably from the sought ones. If small errors due to a rounding of the input values to integers ($\delta_0 \sim 0.001\Theta_{max}$) are introduced into Θ_n, however, then the direct method will become entirely unfeasible at sufficiently small values of ΔFo. Meanwhile, the regularized solution is very close to the exact solution.

In order to evaluate the accuracy of the solution to the reverse heat-conduction problem, as a function of the error in the initial values, the Θ_n values were "perturbed" as follows:

1. $\Theta_{\delta n} = \Theta_n + (-1)^n\delta_0$ is the saw-tooth perturbation;
2. $\Theta_{\delta n} = \Theta_n + (\delta_0/3)\omega_n$ is the perturbation according to a normal distribution of probability densities, on the basis of the "three sigma" rule;
3. $\Theta_{\delta n} = \Theta_n + \delta_0\xi_n$ is the perturbation according to a uniform distribution of probability densities;
4. $\Theta_{\delta n} = \Theta_n \pm \delta_0$ is the constant systematic error.

Here δ_0 denoted the largest possible error, ω_n was a random quantity distributed normally with the mathematical expectation m = 0 and the dispersion D = 1, ξ_n was a random quantity distributed equidensely on the interval [−1, 1].

The first three perturbation modes have yielded results of comparable accuracy. The inaccuracy of $q_\alpha(\tau)$ was somewhat greater with the uniform distribution of Θ_n errors. The results of regularized approximations at $x_1 = 0.6 \cdot 10^{-3}$m for this case are shown in Fig. 2a. With the temperature probe moved away from the body surface to a distance $x_1 = 2.4 \cdot 10^{-3}$m, where $\Theta_{max} \simeq 20$°K and $\Theta \sim 0$ during the initial period (Fo = 0—1), the accuracy of restoring the $q(\tau)$ curve becomes somewhat worse (Fig. 2b), but even now must still be considered close. With the fourth kind of perturbation, the error in determining the thermal flux was approximately equal to the error in the initial temperature values.

Conditions $\bar{q}'(0) = C_1$ and $\bar{q}'(\tau_m) = C_2$ are in many practical cases unknown and must be replaced by so called natural boundary conditions ($C_1 = C_2 = 0$). As a result, around the end points $\tau = 0$ and $\tau = \tau_m$ the regularized approximation deviates from the sought function (Fig. 2a, 3). This deviation becomes larger, as the fluctuating errors in the input data increase. If it is important to reduce the range affected by a priori defined initial and boundary conditions, therefore, one may begin by first smoothing out the input data. Such an approach is justified also in the case where the standard deviation of temperature readings has not been estimated and the choice of α can be based only on the closest proximity of regularized solutions (4), a condition which is very decisive at small values of δ. For smoothing out the input data, one needs algorithms based on the general method of regularization [6].

It would be of interest to evaluate the effectiveness of zeroth-order regularization in solving a reverse heat-conduction problem with the regularizing functional

$$\Phi^{\alpha}[q, \Theta] = \|Aq - \Theta\|^{r}_{L_2} + \alpha\|q\|^{r}_{L_2},$$

where $\|-\|_{L_2}$ denotes a norm in the Hilbert space L_2 of functionals. In this case there is no uniform convergence of regularized solutions. Indeed, it follows from the appropriate Euler equation [1]

$$\frac{1}{\alpha}\int_0^{\tau_m} \bar{K}(\xi, \zeta)\, n(\zeta)d\zeta - \frac{\bar{b}(\xi)}{\alpha} + u(\xi) = 0,$$

where

$$\bar{K}(\xi, \zeta) = \begin{cases} \int_{\xi}^{\tau_m} K(\tau-\xi)\,K(\tau-\zeta)\,d\tau, & 0 \leqslant \zeta \leqslant \xi, \\ \int_{\zeta}^{\tau_m} K(\tau-\xi)\,K(\tau-\zeta)\,d\tau, & \xi \leqslant \zeta \leqslant \tau_m, \end{cases}$$

$$\bar{b}(\xi) = \int_{\xi}^{\tau_m} \Theta(\tau)\,K(\tau, \xi)\,d\tau,$$

that the regularized solutions will become tied at the end to a zero value. Such a method of regularizing the solutions to reverse heat-conduction problems yields satisfactory results, if the boundary function to be restored is close to zero at $\tau = \tau_m$ (Fig. 3).

As many computer-simulated experiments have shown, a sufficiently accurate and reliable restoration of boundary conditions requires the combining of the quasioptimum parameter method (4) with the remainder principle (5) or with condition (6). Moreover, inasmuch as the remainder principle always yields "oversmoothed" results, a much better approximation would be the regularized solution which corresponds to the first local minimum Δ_1 to the left of $\alpha_i (\alpha < \alpha_i)$. We note that, with small errors in the initial temperature values, the choice of closer approximations to the sought boundary function on the basis of condition (4), (5), or (6) leads to almost identical results. For finding the sought approximation, therefore, one may use the quasioptimum parameter method alone and in this method one does not need to know the error in the input data.

In some problems it is worthwhile to improve the accuracy of approximating operator A with $A_{\Delta T}$. This is the case, for example, in problems where the dynamic characteristics of external heat loads change appreciably, also in problems where very accurate results are required from input data given with sufficiently small errors.

A very accurate approximation to the operator which establishes the correspondence between q and Θ becomes the overriding requirement for an effective use of the remainder principle in the second kind of problem, inasmuch as it is implied in this principle that the error of the operator approximation is negligibly smaller than the error in the input data. On this basis, then, we will construct a regularization scheme for solving a reverse heat-conduction problem with the computation analog of the integral equation derived for a piecewise-linear approximation to the sought thermal flux $q(\tau)$ curve [2]:

$$\Theta_n(x_1) = \frac{2\sqrt{a}}{\lambda_0}\sum_{i=1}^{n}\left(F_i^n K_i - f_i^n q_{i-1}\right). \tag{7}$$

Following the just described procedure for obtaining a regularization algorithm, and omitting all intermediate steps, we write down the final result in the form of the system of algebraic equations (boundary conditions for the Euler equation $q'(0) = q'(\tau_m) = 0$):

$$\left(a_{0,0} + \frac{\alpha}{\Delta\tau^2}\right)q_0 + \left(b_{0,1} - \frac{\alpha}{\Delta\tau^2}\right)q_1 + b_{0,2}q_2 + \cdots$$

$$+ b_{0,m-1}q_{m-1} + c_{0,m}q_m = \sum_{n=1}^{m} \varphi_1^n \Theta_{\delta n},$$

$$\left(b_{1,0}-\frac{\alpha}{\Delta\tau^2}\right)q_0+\left(g_{1,1}+\frac{2\alpha}{\Delta\tau^2}\right)q_1+\left(f_{1,2}-\frac{\alpha}{\Delta\tau^2}\right)q_2$$

$$+f_{1,3}q_3+\ldots+f_{1,m-1}q_{m-1}+d_{1,m}q_m=\Phi_1^1\Theta\delta_1+\sum_{i=2}^{m}(\Phi_1^n+\varphi_2^n)\Theta\delta_n,$$

$$b_{2,0}q_0+\left(f_{2,1}-\frac{\alpha}{\Delta\tau^2}\right)q_1+\left(g_{2,2}+\frac{2\alpha}{\Delta\tau^2}\right)q_2+\left(f_{2,3}-\frac{\alpha}{\Delta\tau^2}\right)q_3$$

$$+f_{2,4}q_4+\ldots+f_{2,m-1}q_{m-1}+d_{2,m}q_m$$

$$=\Phi_2^2\Theta_{\delta 2}+\sum_{i=3}^{m}(\Phi_2^n+\varphi_3^n)\Theta_{\delta n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_{m-1,0}q_0+f_{m-1,1}q_1+\ldots+f_{m-1,m-3}q_{m-3}+\left(f_{m-1,m-2}-\frac{\alpha}{\Delta\tau^2}\right)q_{m-2}$$

$$+\left(q_{m-1,m-1}+\frac{2\alpha}{\Delta\tau^2}\right)q_{m-1}+\left(d_{m-1,m}-\frac{\alpha}{\Delta\tau^2}\right)q_m$$

$$=\Phi_{m-1}^{m-1}\Theta_{\delta m-1}+(\Phi_{m-1}^m+\varphi_m^m)\Theta_{\delta m},$$

$$c_{m,0}q_0+d_{m,1}q_1+\ldots+d_{m,m-3}q_{m-3}+d_{m,m-2},\ q_{m-2}$$

$$+\left(d_{m,m-1}-\frac{\alpha}{\Delta\tau^2}\right)q_{m-1}+\left(e_{m,m}+\frac{\alpha}{\Delta\tau^2}\right)q_m=\Phi_m^m\Theta_{\delta m},$$

where

$$a_{0,0}=\sum_{n=1}^{m}\varphi_1^n\varphi_1^n;$$

$$b_{0,k}=b_{k,0}=\Phi_k^k\varphi_1^k+\sum_{n=k+1}^{m}\varphi_1^n(\Phi_k^n+\varphi_{k+1}^n);$$

$$c_{0,m}=\Phi_m^m\varphi_1^m;$$

$$d_{m,k}=d_{k,m}=\Phi_m^m(\Phi_i^m+\varphi_{k+1}^m);$$

$$e_{m,m}=\Phi_m^m\Phi_m^m;$$

$$g_{kk}=\Phi_k^k\Phi_k^k+\sum_{n=k+1}^{m}(\Phi_k^n+\varphi_{k+1}^n)(\Phi_k^n+\varphi_{k+1}^n);$$

$$f_{l,k}=f_{k,l}=\Phi_k^k(\Phi_l^k+\varphi_{l+1}^k)+\sum_{n=k+1}^{m}(\Phi_l^k+\varphi_{l+1}^k)(\Phi_k^n+\Phi_{k+1}^n),$$

$$k=1,2,\ldots,m-1;\ l=1,2,\ldots,m-1;$$

$$\Phi_1^n=\frac{2\sqrt{a\Delta\tau}}{\lambda_0\Delta\tau}F_i^n;\quad \varphi_1^n=-\left(\Phi_i^n+\frac{2\sqrt{a\Delta\tau}}{\lambda_0}f_i^n\right);$$

$$f_i^n=\sqrt{n-p}\,i\Phi^*\left[\frac{x_1}{2\sqrt{a\Delta\tau(n-p)}}\right]\Bigg|_{p=i-1}^{p=i};$$

$$F_i^n=\left[(i-1)\Delta\tau-n\Delta\tau-\frac{x_1^2}{6}\right]f_i^n$$

$$+\frac{\Delta\tau}{3\sqrt{\pi}}\left[(n-p)^{3/2}e^{\frac{-x_1^2}{4a\Delta\tau(n-p)}}\right]\Bigg|_{p=i-1}^{p=i}.$$

Starting the regularization process with relation (7) makes it possible to improve the approximation to the operator A. As a consequence, for obtaining the same accuracy in restoring the thermal flux curve, one may make the $\Delta\tau$ interval wider than in the algorithm based on Eq. (3) and thus reduce the necessary computer time or increase the widest possible $[0, \tau_m]$ interval on which a boundary condition is to be restored.

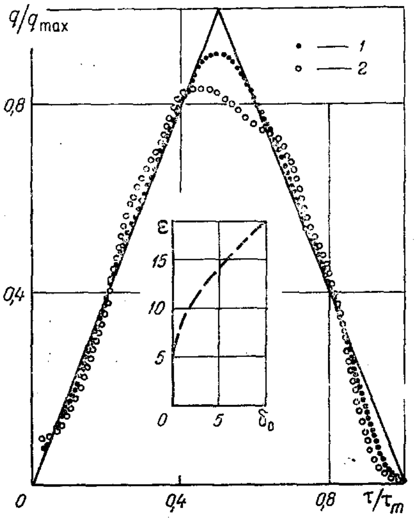

Fig. 3. Zeroth-order regularization for a "triangular" thermal flux $\bar{q}$ (°C) with a uniform distribution of errors in the initial temperature values, with $\Delta \mathrm{Fo} = 0.02$ and $\delta_0 = 0.01\Theta_{max}$ (1), $0.1\Theta_{max}$ (2). Solid line represents the exact solution, dashed line represents the estimated accuracy of restoring the thermal flux curve. Parameters ε and δ both in %.

The last statement is relevant to the "dimensionality problem" concerning the limited memory of modern computers and their inability to solve high-dimensional problems. The dimensionality of a problem is determined by the highest order of the system of algebraic equations (3) or (7), equal to the largest possible number of subdivisions of the time interval ($m_{max} \simeq 70$ for solving Eq. (3) on an M-20 computer).

In practice one often encounters the problem of determining the thermal flux to a plate with one surface insulated where the temperature is measured. For this case there has been derived an integral equation (with a kernel not containing infinite series) in [7] for determining an auxiliary function $g(\tau)$ coupled to $q(\tau)$ through a continuous operator. The first part of this equation represents the derivative of the test temperature with respect to time $d\Theta/d\tau$, the determination of which is a problem in the noncorrective class. For this reason, it is worthwhile to construct the computation algorithm so as to bypass the intermediate step of calculating the derivative.

We use the integral equation in $g(\tau)$ [8], representing it in the form

$$\int_0^\tau g(\xi)\frac{\partial}{\partial\tau}\vartheta(\tau-\xi)\,d\xi = \frac{d\Theta(\tau)}{d\tau},$$

where

$$\vartheta(\tau-\xi) = \Phi^*\left[\frac{b}{2\sqrt{a(\tau-\xi)}}\right] \equiv 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{b}{2\sqrt{a(\tau-\xi)}}} \exp[-\eta^2]\,d\eta.$$

From here we have

$$\int_0^\tau d\tau \int_0^\tau g(\xi)\frac{\partial}{\partial\tau}\vartheta(\tau-\xi)\,d\xi = \Theta(\tau).$$

Changing the order of integration yields

$$\int_0^\tau g(\xi)\,\Phi^*\left[\frac{b}{2\sqrt{a(\tau-\xi)}}\right]d\xi = \Theta(\tau).$$

Using the approximation technique shown in [2], we write

$$\sum_{i=1}^{n} \bar{g}_i \vartheta_i^n = \Theta_n, \quad n = 1, 2, \ldots, m,$$

where

$$\vartheta_i^n = \frac{b^2}{2a}\left[F(Y_i^n) - F(Y_{i-1}^n)\right]; \quad Y_i^n = \frac{b}{2\sqrt{a(\tau_n-\tau_i)}};$$

$$F(Y) = \frac{1}{Y}\left[i\Phi^*(Y) - 0.5\,\frac{\Phi^*(Y)}{Y}\right].$$

Thus, determining the auxiliary function g(τ) falls within the framework of the described regularization procedure according to (3). The relating equation q(g(τ), τ) is

$$q(\tau)=\frac{\lambda_0}{2\sqrt{a\pi}}\int_0^{\tau} g(\xi)\,\frac{1-\exp\left[-\dfrac{b^2}{a(\tau-\xi)}\right]}{\sqrt{\tau-\xi}}\,d\xi.$$

After an approximation we obtain the following computation formula:

$$q_n=\frac{\lambda_0}{\sqrt{a\pi}}\sum_{i=1}^{n}\bar{g}_i\varphi_i^n,\quad n=1,\ 2,\ \ldots,\ m,$$

where

$$\bar{g}_i=\frac{g_i+g_{i-1}}{2};\quad \varphi_i^n=\left\{\sqrt{\tau_n-\xi}-\sqrt{\pi(\tau_n-\xi)}\ i\Phi^*\right.$$
$$\left.\times\left[\frac{b}{\sqrt{a(\tau_n-\xi)}}\right]\right\}_{\xi=\tau_i}^{\xi=\tau_{i-1}}.$$

Solving several model problems of determining the thermal fluxes has shown that the procedure is computationally as efficient as the procedure for a semiinfinite body, except for the somewhat wider "edge effect" region subject to the arbitrary choice of condition $g'(\tau_m) = 0$.

The proposed method of determining the boundary conditions is suitable for a large class of problems involving the simulation of transient thermal modes in the laboratory and in field testing of diverse apparatus.

NOTATION

A	is the integral operator;
$A_{\Delta\tau}$	is the approximating operator;
a	is the thermal diffusivity;
b	is the plate thickness;
g	is the auxiliary function;
q	is the thermal flux;
T	is the temperature;
x_1	is the distance from body surface to temperature probe;
Fo	is the Fourier number;
α	is the regularization parameter;
δ_{L_2}	is the error of input data in the metrics of the Hilbert space of functionals;
ε	is the largest error in restoring the thermal flux curve, with respect to q_{max};
$\Delta\tau$	is the time interval;
Θ	is the model temperature;
λ	is the thermal conductivity;
τ	is the time;
τ_m	is the right-hand boundary of time interval;

$\Delta Fo = a\Delta\tau/x_1^2$.

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